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Higher Loop Anomalies and their Consistency Conditions in Nonlocal Regularization

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Abstract

An algebraic program of computation and characterization of higher loop BRST anomalies is presented. We propose a procedure for disentangling a genuine *local* higher loop anomaly from the quantum dressings of lower loop anomalies. For such higher loop anomalies we derive a local consistency condition, which is the generalisation of the Wess-Zumino condition for the one-loop anomaly. The development is presented in the framework of the field-antifield formalism, making use of a nonlocal regularization method. The theoretical construction is exemplified by explicitly computing the two-loop anomaly of chiral W_3 gravity. We also give, for the first time, an explicit check of the local two–loop consistency condition that is associated with this anomaly.

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1 Introduction

The phenomenon of gauge anomalies [1] –obstructions to the implementation of classical gauge symmetries at quantum level– has been known for quite a long time to deeply relate to the algebraic structure of gauge theories. Its algebraic analysis, initiated in the pioneering work of Becchi, Rouet and Stora [2], has since then been a fruitful tool to determine the viability of the renormalization program for such type of theories [3]. The BRS approach defines an anomaly as an obstruction to the fulfilment of the BRST-Ward identities. At lowest order their characterization is reduced to the algebraic problem of classifying solutions of the Wess-Zumino consistency conditions [4]. Anomaly candidates appear as local cohomology classes at ghost number one of the nilpotent operator δ generating the BRST symmetry [2, 5] of the classical theory.

BRS ideas have later been generalized and further extended by means of (among other approaches) the so-called antibracket-antifield formalism, or Batalin-Vilkovisky formalism [6] or, in short, FA formalism (see [7, 8, 9] for recent reviews). This proposal starts by first organizing the fields required to covariantly quantize the theory and their corresponding BRS sources [10] in a set of fields and antifields $\{\Phi^A,\, \Phi^*_A\}$ viewed as local coordinates of an extended "phase space" $\mathcal{M}$, which is endowed with an odd symplectic structure $(\cdot,\cdot)$ –the antibracket–

$$(X,Y)=\left(\frac{\partial_r X}{\partial\Phi^A}\frac{\partial_l Y}{\partial\Phi^*_A}-\frac{\partial_r X}{\partial\Phi^*_A}\frac{\partial_l Y}{\partial\Phi^A}\right), \tag{1.1}$$

and, for the quantum theory, with a nilpotent second order differential operator Δ

$$\Delta\equiv(-1)^{(A+1)}\frac{\partial_r}{\partial\Phi^A}\frac{\partial_r}{\partial\Phi^*_A}, \qquad \Delta^2\equiv 0.$$

In this geometrical setting, the classical BRST structure of a given gauge theory is encoded in a generalized action $S(\Phi,\Phi^*)$. Its antifield expansion

$$S(\Phi,\Phi^*)=\mathcal{S}(\Phi)+\sum_{n\geq 1}\frac{1}{n!}\Phi^*_{A_1}\ldots\Phi^*_{A_n}R^{A_n\ldots A_1}(\Phi), \tag{1.2}$$

contains as coefficients a suitable gauge-fixed action, $\mathcal{S}(\Phi)$, the BRST symmetry transformations $\delta\Phi^A=R^A(\Phi)$, and higher order structure functions, $R^{A_n\cdots A_1}(\Phi)$ describing the classical BRST structure. The basic equation is the so-called classical master equation

$$(S,S)=0, \tag{1.3}$$

of which S is a proper solution. Due to this equation, S itself is invariant under the generalized BRST symmetry $\hat\delta$ in $\mathcal{M}$ defined by

$$\hat\delta F(\Phi,\Phi^*)\equiv(F,S). \tag{1.4}$$

Quantization of the theory is accomplished by path–integrating over a suitable quantum extension W of the action S (1.2)

$$W=S+\sum_{p\geq 1}\hbar^p M_p, \tag{1.5}$$

where we anticipated the presence of local counterterms to guarantee finiteness of the theory (and possibly to preserve quantum BRST invariance as well). The corresponding effective action $\Gamma(\Phi,\Phi^*)$ incorporates then all loop effects to the classical BRST structure

functions and is thus interpreted, in analogy with S (1.2), as the generating functional of their quantum counterparts. In this way the quantum deformation of the classical BRST structure is naturally described by means of the quantum analog of the classical master equation (1.3) –the (anomalous) Zinn-Justin equation [10]–

$$\frac{1}{2}(\Gamma,\Gamma) = -i\hbar(\mathcal{A}\cdot\Gamma), \tag{1.6}$$

where $(\mathcal{A}\cdot\Gamma)$ stands for the generating functional of 1PI Green functions with one insertion of a composite field $\mathcal{A}$. This composite operator $\mathcal{A}$ parametrizes the departure from the classical BRST structure due to quantum corrections, and is interpreted as the BRST anomaly. The FA formalism gives an explicit expression for this anomaly

$$\mathcal{A} = \left[\Delta W + \frac{i}{2\hbar}(W,W)\right], \tag{1.7}$$

which however is very formal, since the action of the operator Δ on *local* expressions is ill-defined, proportional to $\delta(0)$. Whereas it is possible to obtain a corresponding expression in renormalised perturbation theory using Zimmerman's normal products [11], we will follow in this paper a different method where, on the regularised level (i.e. also before the actual renormalisation) these expressions do make sense because effectively all quantities on which Δ acts will be *nonlocal*.

On the level of the effective action the generalized BRST transformation $\hat{\delta}$ (1.4) is [8]

$$\hat{\delta}_Q\,(\mathcal{B}\cdot\Gamma) \equiv ((\mathcal{B}\cdot\Gamma),\Gamma) = ((\sigma\mathcal{B}+\mathcal{B}\,\mathcal{A})\cdot\Gamma)\,, \tag{1.8}$$

relation which extends to the general anomalous case the result previously obtained in [12] for nonanomalous theories, namely $((\mathcal{B}\cdot\Gamma),\Gamma) = ((\sigma\mathcal{B})\cdot\Gamma)$. In these expressions, the so-called *quantum BRST operator* σ, a possible[1] quantum generalization [13] of the classical BRST transformation (1.4), is formally given in terms of the generalized quantum action W (1.5) by

$$\sigma\mathcal{B} \equiv (\mathcal{B},W) - i\hbar\Delta\mathcal{B}. \tag{1.9}$$

Equation (1.8), together with the FA form of the BRST anomaly (1.7), is one of the cornerstones of the treatment of anomalies in this framework. The operator σ (1.9) contained in it determines a set of local consistency conditions for $\mathcal{A}$ in (1.6). Indeed, a straightforward use of the Jacobi identity for the antibracket leads to the following equation for $(\mathcal{A}\cdot\Gamma)$:

$$((\Gamma,\Gamma),\Gamma) \equiv 0 \;\Rightarrow\; ((\mathcal{A}\cdot\Gamma),\Gamma) \equiv 0, \tag{1.10}$$

and therefore, by (1.8),

$$\sigma\mathcal{A} = 0\,. \tag{1.11}$$

The vanishing of the action of the quantum BRST operator σ on $\mathcal{A}$ is then the formal expression for the consistency condition on the BRST anomaly. If we expand it in powers of $\hbar$ (i.e. a loop-wise expansion), $\mathcal{A} = \sum_{p=1}\hbar^{p-1}\mathcal{A}_p$, we obtain the formal expression for the p-loop consistency conditions

$$(\mathcal{A}_1,S) = 0, \tag{1.12}$$

$$(\mathcal{A}_2,S) + (\mathcal{A}_1,M_1) - i\Delta\mathcal{A}_1 = 0, \tag{1.13}$$

$$(\mathcal{A}_p,S) + \sum_{q=1}^{p-1}(\mathcal{A}_q,M_{p-q}) - i\Delta\mathcal{A}_{p-1} = 0, \qquad p>2. \tag{1.14}$$

[1]The alternative is to include the $\mathcal{A}$-term: then the transformation is nilpotent [9].

The first of these equations is clearly the Wess-Zumino consistency condition for the lowest order part of the anomaly [14, 15]. The rest of the relations, (1.13) and (1.14), are the generalization for higher loop anomalies.

The characterization of one loop anomalies by means of (1.12) and the associated cohomological problem, and its calculation through (1.7), have received considerable attention in recent years. A similar algebraic higher loop analysis, on the other hand, is almost completely missing. In no small measure, this is due to the fact that, although the Field-Antifield formalism computes and characterizes the BRST anomalies from an algebraic point of view in a straightforward manner, the resulting equations remain formal since they involve the action of Δ on local expressions: the equation (1.12) for the one loop part is the only exception to this. In ref. [16] the cohomological question was adressed using condition (1.10), and seemed to lead to the conclusion that all higher loop anomalies would satisfy the Wess-Zumino consistency condition (1.12). This is in disagreement with the formal FA result (1.13) and (1.14), which indicates the presence of extra pieces related with the action of Δ on lower order anomalies[2]. In [11], an all-loop perturbative treatment was set up using well-defined expressions in terms of normal product operators, but the issue of consistency conditions was not addressed. Whereas this treatment showed that it is feasable to set up the quantum theory without introducing the Δ operator, it remains nevertheless true that, formally, this operator is very useful to highlight the structure of the FA quantum theory. On the one-loop level, a Pauli-Villars scheme has been introduced [14] to deal in particular with the action of Δ on a local W in eq. (1.7), avoiding the $\delta(0)$ type singularity, and even with its action on a generic local functional or composite field [18].

In this paper we use the nonlocal regularization method of [19], which was shown in [20] to deal very effectively with Δ, by making sure that it acts only on *non-local* expressions at the regularized level, thus avoiding problems with its definition while keeping it as a powerful tool to investigate the quantum theory. The treatment in [20] fell short of a treatment of higher loop anomalies however, in particular of the consistency conditions that they satisfy.

Our main purpose in this paper is therefore to push the computation and algebraic characterization of higher loop anomalies further. We will show how the regularisation problems of the higher loop consistency equations are overcome in the framework of the nonlocally regularized FA formalism. First we demonstrate how information about genuine, local higher loop anomalies can be extracted from the nonlocally regulated version of eq. (1.7). Actually, they arise from quantum dressings of terms contained in the nonlocally regulated version of the insertion $\mathcal{A}$ (1.7) that are evanescent: they formally disappear when the non-local cutoff is removed but may nevertheless give a finite contribution when the limiting process is treated more accurately. Once their origin is identified, the nonlocally regulated version of the BRST Ward identity (1.6) and of its associated Jacobi identity (1.10) lead to a set of local consistency conditions for them.

We have organized the paper as follows. After summarizing in section 2 the basics of the nonlocally regulated field-antifield formalism as developed in [20], we show and illustrate in section 3 how to obtain the local higher loop anomalies, and in section 4 their consistency conditions. The theoretical part is further supported in section 5 by explicitly computing the two-loop anomaly in the example of chiral W_3 gravity and presenting, for the first time, an explicit and complete check of the local two–loop consistency condition associated with this anomaly. Section 6 summarizes conclusions and perspectives.

[2] In this respect, see ref. [17], where generalized consistency conditions for pure W_3 gravity are obtained by algebraic arguments, not relying on perturbation theory.

2 Nonlocal Regularization in the FA framework

The nonlocally regularized FA formalism [20] arises as an extension and reformulation of the nonlocal regularization method of refs. [19] along the line of the antibracket–antifield formalism, giving sense to both diagramatic and formal algebraic computations in this framework. Roughly speaking, nonlocal regularization starts by reformulating the theory in an alternative but equivalent nonlocal way, in which auxiliary fields completely encode in their loops the divergencies of the original theory. Eliminating these loops, through the elimination of these auxiliary fields by putting them equal to their on–shell value, gets rid of their divergent loops and regularizes the theory. Remarkably, this can be done while preserving the BRST (or gauge) symmetry: the resulting nonlocal, regularized theory, is still invariant classically under a nonlocal, distorted version of the original BRST transformation. Previously ill–defined quantities and manipulations –ill–defined because of the locality of the expressions– such as the expression for the FA BRST anomaly (1.7) or its consistency condition (1.11), acquire a well–defined meaning in this process. Of course, as always, after the regularization step one still has to take into account the renormalization to arrive at the end results. We provide for this by incorporating the inclusion of the necessary counterterms from the start.

In this section we summarize the main ideas of this regularized framework, paving the way for our basic goal, namely to identify the origin of higher loop anomalies in this approach and to explore their consistency conditions.

Consider a FA quantized gauge theory and decompose the proper solution S (1.2) into free and interacting parts

$$S(\Phi,\Phi^*) = F(\Phi) + I_{\rm cl}(\Phi,\Phi^*), \qquad \text{with} \qquad F(\Phi) = \frac{1}{2}\Phi^A F_{AB}\Phi^B, \tag{2.1}$$

and where[3] the classical interaction part $I_{\rm cl}(\Phi,\Phi^*)$ is assumed to be analytic in Φ^A around $\Phi^A = 0$. The quantum action (1.5) has a similar perturbative expansion

$$W = F + I_{\rm cl} + \sum_{p\geq 1} \hbar^p M_p \equiv F + I,$$

in terms of a generalized quantum interaction $I(\Phi,\Phi^*)$ which includes the Λ-dependent counterterms mentioned above.

The cut–off parameter Λ^2 and a smearing operator ε are introduced as follows. First choose a field independent[4] (graded symmetric) operator $(T^{-1})^{AB}$ in such a way that a second order derivative "regulator" $\mathcal{R}^A_B$ arises through the combination

$$\mathcal{R}^A_B = (T^{-1})^{AC} F_{CB}. \tag{2.2}$$

The regulator is intended to provide a momentum cutoff through an ubiquitous smearing operator ε^A_B:

$$\varepsilon^A_B = \exp\left(\frac{\mathcal{R}^A_B}{2\Lambda^2}\right). \tag{2.3}$$

[3] We use de Witt's notation, where the space–time point on which a field depends is included in the index of that field. Summation over this index includes a space–time integral. Functional derivatives with respect to fields or antifields are indicated with a lower or upper index respectively: for example, $W^A_B = \frac{\partial_r\partial_l W}{\partial\Phi^B\partial\Phi^*_A}$ and $W_{AB} = \frac{\partial_r\partial_l W}{\partial\Phi^B\partial\Phi^A}$. This notation will apply throughout this paper, for example for F_{AB} in (2.1), or later in this section in (2.14) and (2.15), except if the indexed object is defined explicitly, as in the following formulas for $\mathcal{R}$, ε and $\mathcal{O}$.

[4] One might relax this restriction if the need arises, but for simplicity we will not do so.

The original phase space $\mathcal{M}$ is now temporarily enlarged with the so-called "shadow" fields and antifields $\{\Psi^A, \Psi^*_A\}$, which have the same statistics as the original fields $\{\Phi^A, \Phi^*_A\}$, extending also the antibracket structure (1.1) in the natural way. For these fields one takes as (minus) the propagator the "shadow propagator" $\mathcal{O}^{AB}$:

$$\mathcal{O}^{AB} = \left(\frac{(\varepsilon^2-1)}{F}\right)^{AB} = \left[\int_0^1 \frac{\mathrm{d}t}{\Lambda^2} \exp\left(t\frac{\mathcal{R}^A_C}{\Lambda^2}\right)\right] (T^{-1})^{CB}. \tag{2.4}$$

The interactions are most conveniently described in terms of a different set of canonical coordinates $\{\Theta^A, \Theta^*_A; \Sigma^A, \Sigma^*_A\}$, related to the original ones by the linear canonical transformation

$$\begin{aligned} \Theta^A &= \Phi^A + \Psi^A, \qquad \Theta^*_A = \left[\Phi^*_B(\varepsilon^2)^B_A + \Psi^*_B(1-\varepsilon^2)^B_A\right], \\ \Sigma^A &= \left[(1-\varepsilon^2)^A_B\Phi^B - (\varepsilon^2)^A_B\Psi^B\right], \qquad \Sigma^*_A = \Phi^*_A - \Psi^*_A. \end{aligned} \tag{2.5}$$

The action is now rewritten as follows: replace in the old free action the original fields Φ^A with the smeared fields $(\varepsilon^{-1})^A_B\Phi^B$, add for the shadow fields a free part with the propagator (2.4) constructed above, and for the interaction terms take the old interaction functional I but change the value of its arguments from $\{\Phi^A, \Phi^*_A\}$ to $\{\Theta^A, \Theta^*_A\}$ (2.5). This yields the auxiliary quantum action

$$\begin{aligned} \tilde{W}(\Phi, \Phi^*, \Psi, \Psi^*) &= F(\varepsilon^{-1}\Phi) - \frac{1}{2}\Psi^A \mathcal{O}^{-1}_{AB}\Psi^B + I(\Theta, \Theta^*) \\ &= W(\Theta, \Theta^*) + \frac{1}{2}\Sigma^A \left[\frac{\mathcal{F}}{\varepsilon^2} + \frac{\mathcal{F}}{(1-\varepsilon^2)}\right]_{AB} \Sigma^B. \end{aligned} \tag{2.6}$$

The remarkable result of [20] is that this process does not interfere with the FA structure, and a fortiori not with the BRST structure of the theory. This clarifies the fact that the gauge symmetry is preserved on the classical level, albeit in a deformed way, which in fact was the principal motivation for the introduction of this variant of non-local regularisation [19] for Yang-Mills theories. In view of the generality of the FA formalism, it also immediately generalises this property to arbitrary gauge theories (open algebras, reducible symmetries, etc.) within the realm of the FA formalism.

All in all, the net result of this preliminary process is an auxiliary perturbative theory completely equivalent to the original one[5] when no external Ψ lines are considered. However, the description of the theory by means of the auxiliary action (2.6) concentrates the loop divergencies solely in loops of the auxiliary fields. In this way, regularization of the theory is achieved by eliminating the closed loops formed with "shadow" lines by hand. This can be implemented by putting the auxiliary fields Ψ equal to their on-shell values and their antifields to zero[6]. The shadow field equations of motion

$$\frac{\partial_r \tilde{W}(\Phi, \Phi^*; \Psi, \Psi^* = 0)}{\partial \Psi^A} = 0 \quad \text{i.e.} \quad \Psi^A = \left(\frac{\partial_r I}{\partial \Phi^B}(\Phi + \Psi, \Phi^*\varepsilon^2)\right)\mathcal{O}^{BA}, \tag{2.7}$$

are to be solved for Ψ, perturbatively both in $\hbar$ and in the coupling constants, and its solution $\bar{\Psi}_q(\Phi, \Phi^*)$ substituted in the auxiliary action (2.6). The result of this second step is the final form of the nonlocal quantum action to be used in (now regularized) perturbative computations:

$$W_\Lambda(\Phi, \Phi^*) \equiv \tilde{W}(\Phi, \Phi^*, \bar{\Psi}_q, \Psi^* = 0). \tag{2.8}$$

[5]For a proof of this equivalence, notice that the sum of the propagators of the Φ and Ψ fields in (2.6) is equal to the original Φ propagator, and see [21], section 7.1.

[6]Even though this is not a canonical transformation, since $\{\Psi, \Psi^*\}$ is not a trivial system, nevertheless this substitution does not invalidate the FA structure since the master equation remains valid [20].

Its loopwise expansion

$$W_\Lambda = S_\Lambda + \sum_{n\geq 1} \hbar^p M_{p,\Lambda},$$

provides the nonlocal regularized versions S_Λ, $M_{p,\Lambda}$, of the classical action (1.2) and of the counterterms M_p in (1.5), respectively.

More generically, to write down the smeared nonlocal functional to be used instead of a given classical functional $\mathcal{F}(\Phi, \Phi^*)$, one performs the substitution

$$(\Phi, \Phi^*) \longrightarrow (\Phi + \bar{\Psi}_q, \Phi^* \varepsilon^2). \tag{2.9}$$

The resulting functional $\mathcal{F}_R(\Phi, \Phi^*) \equiv \mathcal{F}(\Phi + \bar{\Psi}_q, \Phi^* \varepsilon^2)$ will then be denoted generically[7] with a subscript R (for "regulated"). This substitution (2.9) can be interpreted as the nonlocal regularization rule for composite fields: in the (formal) limit $\Lambda \to \infty$ we have that $\varepsilon \to 1$ and $\bar{\Psi}_q \to 0$ [19] (see (2.7) and (2.4)). This prescription was followed, for example, for the interaction terms in the action. Sometimes the actual corresponding quantity to be used in the non-locally regularised theory (which we denote generically by $\mathcal{F}_\Lambda$) is different: an important example is the kinetic term in the action. In fact, from (2.6) it follows that $W_\Lambda - W_R$ consists of the terms arising from the Σ-terms in (2.6) by putting $\Psi = \bar{\Psi}_q$.

The transition from the formal to the regularized theory is generally made by changing the original quantities (S, W, ...) into their nonlocal counterparts (S_Λ, W_Λ, ...), possibly with shifted arguments. Most of the formal relations of the unregularised theory then become true equalities for the regularised theory without any other changes. For instance, the relations characterizing algebraically the regulated BRST symmetry at classical level are now encoded in the set of equations coming from the regulated classical master equation $(S_\Lambda, S_\Lambda) = 0$. This equation is verified by construction [20]. The classical BRST structure is therefore preserved at the regularized level. By the same token, at quantum level the BRST structure and its possible breakdown are described by means of the regulated counterpart of the BRST Ward identity (1.6)

$$\frac{1}{2}(\Gamma_\Lambda, \Gamma_\Lambda) = -i\hbar(\mathcal{A}_{\Lambda R} \cdot \Gamma_\Lambda), \tag{2.10}$$

where Γ_Λ stands for the effective action (1PI) associated to the nonlocally regulated quantum action W_Λ (2.8), and for the insertion we have followed the notational convention just explained. Indeed, the obstruction $\mathcal{A}_{\Lambda R}$ parametrizing this breakdown is still of the form (1.7) with $W \to W_\Lambda$, i.e.

$$\mathcal{A}_{\Lambda R}(\Phi, \Phi^*) = \left[\Delta W_\Lambda + \frac{i}{2\hbar}(W_\Lambda, W_\Lambda)\right](\Phi, \Phi^*) = \mathcal{A}_\Lambda(\Phi + \bar{\Psi}_q, \Phi^* \varepsilon^2), \tag{2.11}$$

where now the action of the operator Δ is well–defined thanks to the nonlocality of W_Λ.

At this point we have regular versions (2.10) and (2.11) of the fundamental equations (1.6) and (1.7) at our disposal. The content of these equations can be explored further using the particular properties following from the construction of the action (2.8). Starting with the insertion $\mathcal{A}_{\Lambda R}$ (2.11), the action of the operator Δ on W_Λ can be explicitly computed. The derivatives of W_Λ (2.8) and $\tilde{W}$ (2.6) are related by

$$\frac{\partial_r W_\Lambda}{\partial \Phi^A} = \left.\frac{\partial_r \tilde{W}}{\partial \Phi^A}\right|_q, \qquad \frac{\partial_l W_\Lambda}{\partial \Phi^*_A} = \left.\frac{\partial_l \tilde{W}}{\partial \Phi^*_A}\right|_q, \tag{2.12}$$

[7] We use the notation $\mathcal{F}_\Lambda$ for the functional of the original fields and antifields that, in the Λ-regulated theory, replaces the functional $\mathcal{F}$. This may already contain the cutoff Λ in certain places. In the regulated theory, these quantities usually appear, in addition, with the shifted arguments (2.9), resulting in $\mathcal{F}_{\Lambda R}(\Phi, \Phi^*) = \mathcal{F}_\Lambda(\Phi + \bar{\Psi}_q, \Phi^* \varepsilon^2)$. It is important to keep in mind the distinction between $\mathcal{F}_\Lambda$, $\mathcal{F}_{\Lambda R}$, and $\mathcal{F}_R$.

where the "q"-restriction means on the surface $\{\Psi = \bar{\Psi}_q(\Phi, \Phi^*), \Psi^* = 0\}$. This yields

$$\Delta W_\Lambda(\Phi, \Phi^*) = \left[W^A_B\,(\delta_\Lambda)^B_C\,(\varepsilon^2)^C_A\right](\Phi + \bar{\Psi}_q, \Phi^*\varepsilon^2) \equiv \Omega_W(\Phi + \bar{\Psi}_q, \Phi^*\varepsilon^2), \tag{2.13}$$

where the matrix $(\delta_\Lambda)^A_B$ is defined as

$$(\delta_\Lambda)^A_B = \frac{\partial_r(\Phi + \bar{\Psi}_q)^A}{\partial \Phi^B} = \mathcal{K}^{AC}(\mathcal{O}^{-1})_{CB} = \left(\delta^A_B - \mathcal{O}^{AC} I_{CB}\right)^{-1}, \tag{2.14}$$

in terms of an operator $\mathcal{K}^{AB}$ whose inverse is given by

$$\mathcal{K}^{-1}_{AB} = (\mathcal{O}^{-1})_{AB} - I_{AB}\,. \tag{2.15}$$

If we compare this expression with the formal computation of ΔW, which would give

$$\Delta W = \frac{\partial_r \partial_l W}{\partial \Phi^B \partial \Phi^*_A}\delta^B_A = W^A_B \delta^B_A, \tag{2.16}$$

we can view the regularised expression in (2.13) as resulting from (2.16) in two steps. The first is special, and consists of inserting the extra factor $\delta_\Lambda \varepsilon^2$ (instead of a delta-function) which may be viewed as smearing out the Δ operator. This results in a quantity we call $\Omega_W(\Phi, \Phi^*)$, which depends on Λ explicitly through the presence of ε^2 and $\mathcal{O}$. The second is to change the fields and antifields in the argument of the functional with the substitution (2.9), the generic regularisation step.

Analogous manipulations allow to rewrite the second term in (2.11) as

$$(W_\Lambda, W_\Lambda)(\Phi, \Phi^*) = (\tilde{W}, \tilde{W})(\Theta, \Theta^*)\Big|_q = (W, W)(\Phi + \bar{\Psi}_q, \Phi^*\varepsilon^2),$$

and expression (2.11) as

$$\mathcal{A}_{\Lambda R}(\Phi, \Phi^*) = \left[\Omega_W + \frac{i}{2\hbar}(W, W)\right](\Phi + \bar{\Psi}_q, \Phi^*\varepsilon^2) \equiv \mathcal{A}_\Lambda(\Phi + \bar{\Psi}_q, \Phi^*\varepsilon^2). \tag{2.17}$$

Assuming that we are working with a renormalisable theory, the validity of the regularization method requires the existence of suitable counterterms M_p cancelling the possible divergencies arising, for example, in the computation of $\Omega_W(\Phi, \Phi^*)$. As stated in the beginning, these counterterms have been assumed to be present in W from the start. Taking the formal $\Lambda \to \infty$ limit of (2.17) then results in a finite local functional $\bar{\mathcal{A}}(\Phi, \Phi^*)$

$$\bar{\mathcal{A}}(\Phi, \Phi^*) = \lim_{\Lambda^2 \to \infty} \mathcal{A}_\Lambda(\Phi + \bar{\Psi}_q, \Phi^*\varepsilon^2) = \lim_{\Lambda^2 \to \infty}\left[\Omega_W + \frac{i}{2\hbar}(W, W)\right](\Phi, \Phi^*). \tag{2.18}$$

One would now be tempted to consider this local functional as the BRST anomaly produced by this regularization procedure and, after the regularisation step $\bar{\mathcal{A}} \to \bar{\mathcal{A}}_R$, to use it as insertion, instead of $\mathcal{A}_{\Lambda R}$ (2.17), in the Zinn-Justin equation (2.10). In fact, a cursory glance confirms that the consistent one–loop anomaly is then correctly reproduced by the tree level part. However, a closer investigation [20] reveals that the contributions to (2.18) of higher order in $\hbar$ are incomplete: they reproduce correctly only the one–loop parts of the contributions to the anomaly generated by the counterterms M_p of (1.5), but *not* the genuine, local higher loop anomalies. For this reason it was conjectured in [20] that the master equation might be incomplete.

Nevertheless, if the regularisation scheme used here is correct, the regulated BRST Ward identity (2.10) must already contain *all* the information about higher loop anomalies. In other words, the quantum dressing of the insertion $\mathcal{A}_{\Lambda R}$ in the regulated theory, $(\mathcal{A}_{\Lambda R} \cdot \Gamma_\Lambda)$, should generate both the (nonlocal) loop corrections to the lower order anomalies and the local, genuine higher loop anomalies, which seem to be absent from $\bar{\mathcal{A}}$. In the next section we therefore analyse in detail the $\Lambda \to \infty$ limiting process.

3 Higher Loop Anomalies in Nonlocal Regularization

To investigate this discrepancy, that the genuine higher loop anomalies do not to show up clearly in (2.18), we now investigate in detail the limiting process by which a finite local expression is obtained for the anomaly $\mathcal{A}$.

The one-loop anomaly functional is computed as [20]

$$\mathcal{A}_1 = \lim_{\Lambda^2\to\infty} \left[\Omega_S + i(M_1, S)\right], \tag{3.1}$$

where Ω_S is the lowest order part in the loopwise expansion of Ω_W (2.13)

$$\Omega_S = \left[S^A_B(\delta^{(0)}_\Lambda)^B_C(\varepsilon^2)^C_A\right], \tag{3.2}$$

and $\delta^{(0)}_\Lambda$ is the classical part of δ_Λ (2.14)

$$(\delta^{(0)}_\Lambda)^A_B = \left(\delta^A_B - \mathcal{O}^{AC}(I_{\rm cl})_{CB}\right)^{-1} = \delta^A_B + \sum_{n\geq 1}\left(\mathcal{O}^{AC}(I_{\rm cl})_{CB}\right)^n. \tag{3.3}$$

To compute the limit of Ω_S, it may be necessary to subtract a "divergent" part, typically diverging as a power of Λ: this is done through a corresponding M_1 via the (M_1, S) term in (3.1). Then the $\Lambda \to \infty$ limit can be taken, and yields a local Λ-independent functional of the fields Φ and Φ^*.

This functional may now subsequently be used as an insertion in (2.10) to investigate the next order. Of course one can not just insert $\mathcal{A}_1$ as it stands (which would give divergent results, even at finite Λ), but rather the corresponding regularised functional ($\mathcal{A}_1 \to \mathcal{A}_{1R}$, see (2.9)); this regularised expression is used in a regularised one–loop computation (using W_Λ and Γ_Λ). The result of this process, after the $\Lambda \to \infty$ limit is taken, can be denoted $(\mathcal{A}_1 \cdot \Gamma)$. In this way one finds

$$\lim_{\Lambda^2\to\infty}(\mathcal{A}_{\Lambda R}\cdot\Gamma_\Lambda) = \mathcal{A}_1 + \hbar\left[\mathcal{A}_2 + (\mathcal{A}_1\cdot\Gamma_1)\right] + O(\hbar^2), \tag{3.4}$$

where $(\mathcal{A}_1 \cdot \Gamma_1)$ is the one–loop piece of $(\mathcal{A}_1 \cdot \Gamma)$, obtained in the way described above, and $\mathcal{A}_2$ is simply the limit of the remaining terms at that order. Had we used instead the regulated version of $\bar{\mathcal{A}}$ (2.18), $\bar{\mathcal{A}}_R$, as insertion, the result would have been

$$\lim_{\Lambda^2\to\infty}(\bar{\mathcal{A}}_R\cdot\Gamma_\Lambda) = \mathcal{A}_1 + \hbar\left[\bar{\mathcal{A}}_2 + (\mathcal{A}_1\cdot\Gamma_1)\right] + O(\hbar^2), \tag{3.5}$$

with $\bar{\mathcal{A}}_2$ encoding only the contributions to the complete two–loop anomaly $\mathcal{A}_2$ generated by the counterterms M_p, $p = 1, 2$, of (1.5), but *not* the genuine, local two–loop anomaly.

The reason for the discrepancy between (3.4) and (3.5) can now be understood as follows. Formally the difference $\mathcal{A}_\Lambda(\Phi, \Phi^*) - \bar{\mathcal{A}}(\Phi, \Phi^*)$ vanishes when one takes the $\Lambda \to \infty$ limit, typically as an inverse power of Λ times a local operator, and can therefore properly be called "evanescent"[8]. One might therefore expect that in the construction above the functional $\mathcal{A}_2 - \bar{\mathcal{A}}_2$, which corresponds to this difference, would vanish. However, in the regularised theory, the quantum corrections to some matrix elements of the local operator $\mathcal{A}_\Lambda - \bar{\mathcal{A}}$ may diverge. More accurately, since in the present context no divergences are present at finite Λ, the loops behave as positive powers for $\Lambda \to \infty$. This divergence may conspire with the formal vanishing of $\mathcal{A}_\Lambda - \bar{\mathcal{A}}$ to give a *finite non-vanishing* result[9]. As a

[8] "likely to vanish" [22].

[9] A similar behaviour was observed by Gervais and Jevicky in [23], on implementing point canonical transformations in a time discretized version of the quantum mechanical path integral.

consequence, when using $\bar{\mathcal{A}}$ (2.18) as insertion in (3.5), this nonvanishing contribution will not be included in the quantum correction to lower order anomaly insertions, and therefore has to be added separately, as a new insertion, in order to reproduce some of the effects of the primary insertion $\mathcal{A}_\Lambda$, which otherwise would be lost. In other words, proceeding in this way, these quantum corrections to evanescent pieces are *effectively* replaced by their effect: this turns out to be the source of the *genuine* higher loop anomalies in this approach.

The steps traced above to order $\hbar$ can be retraced for the higher orders as well. At each order, loop corrections of the lower orders appear, but also additional local terms. It is crucial to realise that the anomaly functional

$$\mathcal{A} = \mathcal{A}_1 + \hbar\mathcal{A}_2 + \dots \tag{3.6}$$

obtained in this way as a power series in $\hbar$ does *not* reproduce the functional $\mathcal{A}_\Lambda$. Instead, it *does* reproduce, by the very definition of the successive terms, all the effects of the $\mathcal{A}_\Lambda$ insertion upon taking the $\Lambda \to \infty$ limit. Namely, although $\mathcal{A}_\Lambda$ and $\mathcal{A}$ differ by terms with inverse powers of Λ, the extra local terms present in $\mathcal{A}$ and not in $\mathcal{A}_\Lambda$ are adjusted by construction so that they reproduce the finite effects of this difference: these arise from loop divergences. In this way, when using the regulated counterpart $\mathcal{A}_R$ of the anomaly functional $\mathcal{A}$ (3.6) as insertion in the regulated theory

$$\mathcal{A}_R(\Phi, \Phi^*) \equiv \left[\mathcal{A}_1 + \hbar\mathcal{A}_2 + O(\hbar^2)\right](\Phi + \bar{\Psi}_q, \Phi^*\varepsilon^2),$$

it correctly reproduces, in the $\Lambda \to \infty$ limit, the results of inserting the quantity $\mathcal{A}_\Lambda$ (2.17), i.e.

$$\lim_{\Lambda^2\to\infty} (\mathcal{A}_{\Lambda R} \cdot \Gamma_\Lambda) = \lim_{\Lambda^2\to\infty} (\mathcal{A}_R \cdot \Gamma_\Lambda) = \mathcal{A}_1 + \hbar\left[\mathcal{A}_2 + (\mathcal{A}_1 \cdot \Gamma_1)\right] + O(\hbar^2). \tag{3.7}$$

For the sake of definiteness, we now derive a closed expression for the genuine two-loop anomaly: it will be clear that the process can be repeated for the higher orders, but the two-loop case suffices to explain the idea. The one-loop corrections to the insertion $\mathcal{A}_\Lambda$ (2.17) then suffice. Higher order quantum dressings of this insertion (2.17) are given, order by order in perturbation theory, by the well-known expression [24]

$$(\mathcal{A}_{\Lambda R} \cdot \Gamma_\Lambda) =: \exp\left\{\frac{i}{\hbar}\sum_{n=2}^{\infty}\frac{(-i\hbar)^n}{n!} G_\Lambda^{A_n\dots A_1}\frac{\partial_l}{\partial\Phi^{A_1}}\cdots\frac{\partial_l}{\partial\Phi^{A_n}}\right\} : \mathcal{A}_{\Lambda R}, \tag{3.8}$$

with the colons ": :" indicating that functional derivates act only on $\mathcal{A}_{\Lambda R}$ and where $G_\Lambda^{A_n\dots A_1}$ stand for the connected correlation functions

$$G_\Lambda^{A_n\dots A_1} = \left\{\frac{\partial_l}{\partial J_{A_n}}\cdots\frac{\partial_l}{\partial J_{A_1}}\left(\frac{i}{\hbar}\ln Z_\Lambda[J,\Phi^*]\right)\right\}\Bigg|_{J=J(\Phi,\Phi^*)},$$

associated with our nonlocally regulated action W_Λ (2.8). Up to two–loops the differential operator in (3.8) is given by

$$1 + \frac{i\hbar}{2}(S_\Lambda^{-1})^{AB}\frac{\partial_l}{\partial\Phi^B}\frac{\partial_l}{\partial\Phi^A} + O(\hbar^2) \equiv 1 + \hbar Q_1 + O(\hbar^2), \tag{3.9}$$

where the "complete" propagator $(S_\Lambda^{-1})^{AB}$ is the inverse of the hessian of the classical part S_Λ of W_Λ, $(S_\Lambda)_{AB} = \frac{\partial_l\partial_r S_\Lambda}{\partial\Phi^A\partial\Phi^B}$. The loop expansion of the insertion $\mathcal{A}_\Lambda$ (2.17) up to second order reads

$$\mathcal{A}_{\Lambda R}(\Phi, \Phi^*) = \left[\mathcal{A}_{1\Lambda} + \hbar\mathcal{A}_{2\Lambda} + O(\hbar^2)\right](\Phi + \bar{\Psi}_q, \Phi^*\varepsilon^2),$$

with the functionals $\mathcal{A}_{1\Lambda}$, $\mathcal{A}_{2\Lambda}$ given by [20]

$$\begin{aligned}\mathcal{A}_{1\Lambda} &= [\Omega_S + i(M_1, S)]\,,\\ \mathcal{A}_{2\Lambda} &= \left[\Omega_{M_1} + \frac{i}{2}(M_1, M_1) + i(M_2, S)\right],\end{aligned}$$

in terms of the zero-th order Ω_S (3.2) and the first order Ω_{M_1} in the loopwise expansion of Ω_W (2.13)[10].

Insertion of these expansions in (3.8) gives the following form of the generating functional of the 1PI diagrams with one insertion of $\mathcal{A}_\Lambda$:

$$(\mathcal{A}_{\Lambda R}\cdot\Gamma_\Lambda) = \mathcal{A}_{1\Lambda}(\Phi+\bar\Psi_q, \Phi^*\varepsilon^2) + \hbar\left[\mathcal{A}_{2\Lambda}(\Phi+\bar\Psi_q, \Phi^*\varepsilon^2) + Q_1\mathcal{A}_{1\Lambda}(\Phi+\bar\Psi_q, \Phi^*\varepsilon^2)\right] + O(\hbar^2). \tag{3.10}$$

Comparing this with eq. (3.4), the expression for the one-loop anomaly provided by non-local regularization is immediatley recognized:

$$\mathcal{A}_1 = \lim_{\Lambda^2\to\infty}\mathcal{A}_{1\Lambda}(\Phi+\bar\Psi_q, \Phi^*\varepsilon^2) = \left\{\lim_{\Lambda^2\to\infty}[\Omega_S + i(M_1,S)]\right\}(\Phi,\Phi^*), \tag{3.11}$$

and agrees with the form (3.1) suggested by the prescription (2.18).

We now continue the investigation at two loop order. There is of course the naively expected contribution from the term $\mathcal{A}_{2\Lambda}$ in (3.10). This is associated with the contribution to the complete two–loop anomaly due to the addition of counterterms M_1 and M_2. In addition there is the one–loop correction to the one–loop anomaly of the regularised theory $\mathcal{A}_{1\Lambda}$:

$$\begin{aligned}Q_1\mathcal{A}_{1\Lambda}(\Phi+\bar\Psi_q, \Phi^*\varepsilon^2) &= \frac{i}{2}\left\{(\varepsilon^2)^A_B(S^{-1})^{BC}\left[(\mathcal{A}_{1\Lambda})_{CD}(\delta^{(0)}_\Lambda)^D_A\right.\right. \qquad (3.12)\\ &+ \left.\left.(-1)^{(D+1)C}\frac{\partial_r\mathcal{A}_{1\Lambda}}{\partial\Phi^D}\frac{\partial_l(\delta^{(0)}_\Lambda)^D_A}{\partial\Phi^C}\right]\right\}(\Phi+\bar\Psi_q, \Phi^*\varepsilon^2) + O(\hbar)\,,\end{aligned}$$

where the $O(\hbar)$ results from replacing δ_Λ by $\delta^{(0)}_\Lambda$ (see (2.14) and (3.3)) and use has been made of the classical analogs of (2.12).

The limit $\Lambda^2\to\infty$ of the one–loop contribution in (3.10) yields both the one–loop dressings $(\mathcal{A}_1\cdot\Gamma_1)$ to the one-loop anomaly $\mathcal{A}_1$ (3.1) *and* the genuine, local two–loop anomaly $\mathcal{A}_2$, i.e.

$$\lim_{\Lambda^2\to\infty}\left[\mathcal{A}_{2\Lambda}(\Phi+\bar\Psi_q, \Phi^*\varepsilon^2) + (\mathcal{A}_{1\Lambda}(\Phi+\bar\Psi_q, \Phi^*\varepsilon^2)\cdot\Gamma_1)\right] = \mathcal{A}_2 + (\mathcal{A}_1\cdot\Gamma_1).$$

We repeat how the last term in this expression is to be computed: first the one–loop anomaly functional is obtained by the limiting process in the previous step, see (3.11); this expression is regularised ($\mathcal{A}_{1R}$), see (2.9); this regularised expression is used in a regularised one–loop computation (using W_Λ and Γ_Λ); finally, the $\Lambda\to\infty$ limit is taken. From (3.8) and (3.9) we can compute this one–loop dressing as

$$(\mathcal{A}_1\cdot\Gamma_1) = \lim_{\Lambda^2\to\infty}\left[Q_1\mathcal{A}_1(\Phi+\bar\Psi_q, \Phi^*\varepsilon^2)\right].$$

This results in the following expression for the genuine, local two–loop anomaly:

$$\mathcal{A}_2 = \lim_{\Lambda^2\to\infty}\left[\mathcal{A}_{2\Lambda}(\Phi+\bar\Psi_q, \Phi^*\varepsilon^2) + Q_1\left((\mathcal{A}_{1\Lambda}-\mathcal{A}_1)(\Phi+\bar\Psi_q, \Phi^*\varepsilon^2)\right)\right]. \tag{3.13}$$

[10]Note the presence of the cutoff in these expressions: see footnote 7 on page 6.

We recall that the $\mathcal{A}_{2\Lambda}$ term is due to the presence of counterterms. The remaining terms are evanescent. Nevertheless, checking the balance of inverse powers of Λ (generically contained in evanescent functionals) with the positive powers arising from the loop–divergence, one concludes that in some circumstances they have a finite limit. This is, in this framework, the origin of the two–loop anomaly.

The finite limit of the second term in (3.13) is actually a *local* functional. This corresponds to the locality of the divergences of quantum field theory: it arises from such a divergence. This fact allows, just as in the one–loop case, considerable simplifications in the actual computations by taking into account rather general features of the theory, like the form of the interaction or dimensional analysis. Therefore, equation (3.13) is not only a perfectly legitimate expression for the two–loop anomaly, but can also be used to actually compute it. Rather than elaborate on the circumstances that may give rise to a non–zero result, we will illustrate the mechanism at work, for chiral W_3 gravity, in section 5.

4 Higher Loop Consistency Conditions

The computation of one–loop anomalies is greatly facilitated by the fact that they satisfy a consistency condition, the celebrated Wess–Zumino condition [4]. In many cases this condition allows only a specific functional form as solution, thus reducing the actual computation of the anomaly to the computation of a coefficient. Having learned how to extract information on higher loop BRST anomalies from the expression of the regulated BRST Ward identity (2.10), we now set out to derive the corresponding consistency conditions for these higher loop anomalies.

In the present regularisation framework, formal expressions become true equalities after substituting for the local functionals their nonlocally regulated counterparts. The regulated version of the original FA consistency condition (1.11), $\sigma\mathcal{A} = 0$, can thus be formulated by replacing $\mathcal{A}$ by its regulated version $\mathcal{A}_{\Lambda R}$ (2.17), i.e. $\sigma\mathcal{A}_{\Lambda R} = 0$. This is simply a consequence of the Jacobi identity for the regulated effective action Γ_Λ

$$((\sigma\mathcal{A}_{\Lambda R}) \cdot \Gamma_\Lambda) = ((\mathcal{A}_{\Lambda R} \cdot \Gamma_\Lambda), \Gamma_\Lambda) = \frac{i}{2\hbar}((\Gamma_\Lambda, \Gamma_\Lambda), \Gamma_\Lambda) \equiv 0\,. \tag{4.1}$$

Although this is an identity, it also implies a restriction on the possible form of the anomaly, as we now explain. Indeed, taking into account our previous result (3.7) and the finiteness of all the limits involved, the following chain of equalities is then seen to hold

$$\begin{aligned}
\lim_{\Lambda^2\to\infty}((\sigma\mathcal{A}_R + \mathcal{A}_R\mathcal{A}_{\Lambda R}) \cdot \Gamma_\Lambda) &= \lim_{\Lambda^2\to\infty}((\mathcal{A}_R \cdot \Gamma_\Lambda), \Gamma_\Lambda) = \\
\left(\lim_{\Lambda^2\to\infty}(\mathcal{A}_R \cdot \Gamma_\Lambda), \lim_{\Lambda^2\to\infty}\Gamma_\Lambda\right) &= \left(\lim_{\Lambda^2\to\infty}(\mathcal{A}_{\Lambda R} \cdot \Gamma_\Lambda), \lim_{\Lambda^2\to\infty}\Gamma_\Lambda\right) = \\
\lim_{\Lambda^2\to\infty}((\mathcal{A}_{\Lambda R} \cdot \Gamma_\Lambda), \Gamma_\Lambda) = 0.
\end{aligned} \tag{4.2}$$

However, whereas (4.1) is an identity that is valid even before taking the $\Lambda \to \infty$ limit, this is no longer true for (4.2). In particular it should be noted that the limit of the insertion $(\sigma\mathcal{A}_R + \mathcal{A}_R\mathcal{A}_{\Lambda R})$ by itself *does not vanish*! The reason is in fact the same as that giving rise to the higher loop anomaly: one can not neglect the effect of evanescent terms on loop diagrams, since their quantum dressing may have a non–vanishing limit. One should therefore keep equation (4.2) as it stands.

The content of (4.2) can be disentangled using a loop-wise expansion. To compute the antibracket in $\sigma\mathcal{A}_R$, namely $(\mathcal{A}_R, W_\Lambda)$, it is expedient to revert to the variables Θ, Θ^*

in intermediate steps: the canonical character of the transformation (2.5) simplifies the computation. With the notation $\tilde{\mathcal{A}}(\Phi,\Phi^*,\Psi,\Psi^*)=\mathcal{A}(\Theta,\Theta^*)$ one finds (for $\tilde{W}$ see (2.6))

$$\begin{aligned}
(\bar{\Psi}^A_q, W_\Lambda)(\Phi,\Phi^*) &= \left.\frac{\partial_l \tilde{W}}{\partial \Psi^*_A}\right|_q + \left[\mathcal{K}^{AB}\frac{\partial_l}{\partial\Phi^B}\frac{1}{2}(W,W)\right](\Phi+\bar{\Psi}_q,\Phi^*\varepsilon^2),\\
(\mathcal{A}_R, W_\Lambda)(\Phi,\Phi^*) &= \left.(\tilde{\mathcal{A}},\tilde{W})\right|_q + \left[\frac{\partial_r\mathcal{A}}{\partial\Phi^A}\,\mathcal{K}^{AB}\,\frac{\partial_l}{\partial\Phi^B}\frac{1}{2}(W,W)\right](\Phi+\bar{\Psi}_q,\Phi^*\varepsilon^2),\\
\left.(\tilde{\mathcal{A}},\tilde{W})\right|_q &= (\mathcal{A},W)(\Theta,\Theta^*)|_q = (\mathcal{A},W)(\Phi+\bar{\Psi}_q,\Phi^*\varepsilon^2).
\end{aligned}$$

The antibracket now becomes[11]

$$(\mathcal{A}_R, W_\Lambda)(\Phi,\Phi^*) = \left[(\mathcal{A},W) + \frac{\partial_r\mathcal{A}}{\partial\Phi^A}\,\mathcal{K}^{AB}\,\frac{\partial_l}{\partial\Phi^B}\frac{1}{2}(W,W)\right](\Phi+\bar{\Psi}_q,\Phi^*\varepsilon^2). \tag{4.3}$$

The explicit expression for $\Delta\mathcal{A}_R$, on the other hand, computed along similar lines to that of ΔW_Λ in section 2, results in

$$\Delta\mathcal{A}_R(\Phi,\Phi^*) \equiv \left(\Omega^{(i)}_{\mathcal{A}} + \Omega^{(ii)}_{\mathcal{A}}\right)(\Phi+\bar{\Psi}_q,\Phi^*\varepsilon^2), \tag{4.4}$$

with the functionals $\Omega^{(i,ii)}_{\mathcal{A}}$ given by

$$\begin{aligned}
\Omega^{(i)}_{\mathcal{A}} &\equiv (-1)^{(A+1)}\left[\mathcal{A}^A_B(\delta_\Lambda)^B_C(\varepsilon^2)^C_A + W^A_B\,(\delta_\Lambda(\mathcal{O}\mathcal{A})\delta_\Lambda)^B_C\,(\varepsilon^2)^C_A\right], \\
\Omega^{(ii)}_{\mathcal{A}} &\equiv \frac{\partial_r\mathcal{A}}{\partial\Phi^A}\,\mathcal{K}^{AB}\,\frac{\partial_l\Omega_W}{\partial\Phi^B},
\end{aligned} \tag{4.5}$$

where $(\mathcal{O}\mathcal{A})^A_B = \mathcal{O}^{AC}\mathcal{A}_{CB}$ and for $\mathcal{K}$ see (2.15).

Putting together (4.4) and (4.3) one gets

$$(\sigma\mathcal{A}_R + \mathcal{A}_R\mathcal{A}_{\Lambda R})(\Phi,\Phi^*) = \left\{(\mathcal{A},W) - i\hbar\Omega^{(i)}_{\mathcal{A}} - i\hbar\frac{\partial_r\mathcal{A}}{\partial\Phi^A}\,\mathcal{K}^{AB}\,\frac{\partial_l\mathcal{A}_\Lambda}{\partial\Phi^B} + \mathcal{A}\mathcal{A}_\Lambda\right\}(\Phi+\bar{\Psi}_q,\Phi^*\varepsilon^2).$$

The antibracket and the $\Omega^{(i)}_{\mathcal{A}}$ term in this expression may then be seen as the regularised version of the naively expected terms. The other terms

$$\left(\mathcal{A}\,\mathcal{A}_\Lambda - i\hbar\frac{\partial_r\mathcal{A}}{\partial\Phi^A}\,\mathcal{K}^{AB}\,\frac{\partial_l\mathcal{A}_\Lambda}{\partial\Phi^B}\right)(\Phi+\bar{\Psi}_q,\Phi^*\varepsilon^2), \tag{4.6}$$

are absent from the naive expressions, for the simple reason that they would vanish identically because of fermi statistics if we would not distinguish between $\mathcal{A}$ and $\mathcal{A}_\Lambda$. Their contribution to the consistency condition is in fact of order $\hbar^2$, and therefore only comes into play when considering the condition for $\mathcal{A}_3$ or higher. The reason is that the product term does not generate tree level or $O(\hbar)$ contributions to the 1PI functional, whereas the second term, with an explicit factor $\hbar$, is evanescent and only contributes through quantum dressings. We will limit ourselves in the rest of this section to the first non-trivial consistency condition, that for the two-loop anomaly, so we do not consider (4.6)

[11] One notes that the process of replacing functionals F,G with their regulated version F_R,G_R does not commute with taking antibrackets: $(F_R,G_R)\neq(F,G)_R$. For the special case of S_Λ however (i.e. W_Λ without the counterterms), one does obtain a simple formula, $(F_R,S_\Lambda)=(F,S)_R$, as can be read off from (4.3), since this equation is in fact valid for arbitrary functionals $\mathcal{A}$.

further. The power series expansion can then be carried out in much the same way as in the previous section. The result is

$$\begin{aligned}\Big((\mathcal{A},W)-i\hbar\Omega^{(i)}_{\mathcal{A}}\Big)\,(\Phi+\bar{\Psi}_q,\Phi^*\varepsilon^2) &= \\ \big\{(\mathcal{A}_1,S)+\hbar\,[(\mathcal{A}_2,S)+(\mathcal{A}_1,M_1)-i\Omega_{\mathcal{A}_1}]+O(\hbar^2)\big\}\,(\Phi+\bar{\Psi}_q,\Phi^*\varepsilon^2).\end{aligned} \tag{4.7}$$

Here $\Omega_{\mathcal{A}_1}$ is the lowest order term in the loop-wise expansion of $\Omega^{(i)}_{\mathcal{A}}$ (4.5)

$$\Omega_{\mathcal{A}_1}=(-1)^{(A+1)}\left[(\mathcal{A}_1)^A_B(\delta^{(0)}_\Lambda)^B_C(\varepsilon^2)^C_A+S^A_B\left(\delta^{(0)}_\Lambda(\mathcal{O}\mathcal{A}_1)\delta^{(0)}_\Lambda\right)^B_C(\varepsilon^2)^C_A\right], \tag{4.8}$$

the notational convention of footnote 3 has been used, and $\delta^{(0)}_\Lambda$ is given by (3.3). The lowest order term in this expression should vanish according to condition (4.2)

$$(\mathcal{A}_1,S)=0,$$

which is the usual Wess–Zumino (one–loop) consistency condition. This same equation plays an important role in simplifying the two–loop anomaly condition, since it implies that when inserting (4.7) in Γ_Λ, equation (4.2), no quantum corrections have to be considered. As a consequence, at this order, the potentially nonlocal (4.2) turns into a *local* equation, which is the final expression for the consistency condition on the genuine two–loop anomaly:

$$\left\{(\mathcal{A}_2,S)+\lim_{\Lambda^2\to\infty}[(\mathcal{A}_1,M_1)-i\Omega_{\mathcal{A}_1}]\right\}(\Phi,\Phi^*)=0. \tag{4.9}$$

The remaining limit is to be understood as before in the one–loop case: the two terms separately can diverge as $\Lambda\to\infty$. There is a difference with (3.1) however. In that case, if a divergence is present, it imposes that M_1 be chosen such that the cancellation occurs and the difference remains finite. Here, M_1 has already been fixed, precisely by that one–loop criterion, so the cancellation in (4.9) should be automatic. Although ultimately this is implied by the finiteness of Γ if the theory is renormalisable, it would be nice to have a more direct proof of this fact from the explicit expressions (4.8) and (3.1).

Summarizing, we have verified that in this nonlocally regulated framework not only the one–loop but also the two–loop anomalies are constrained by a *local* consistency condition, (4.9), which is very similar in form to that obtained in the unregulated formulation, (1.13).

Consistency conditions for higher loop anomalies can now be obtained as well by repeating the procedure demonstrated above. Although the analysis of their explicit form would require a considerable amount of work, it may be inferred from (4.2) that they should also share with (4.9) the generic local form

$$(\mathcal{A}_p,S)-\mathcal{U}_p(\mathcal{A}_1,\ldots,\mathcal{A}_{p-1};S,M_1,\ldots,M_{p-1})=0,\qquad p\geq 2. \tag{4.10}$$

Potential nonlocalities arising in the process should again vanish as a consequence of lower order consistency conditions. Equation (4.10) differs from the standard Wess-Zumino consistency condition by the presence of extra inhomogenities $\mathcal{U}_p$. The cohomological characterization of higher loop parts of the BRST anomaly by means of eqs. (4.10) is therefore different from the lowest order anomaly. The solutions of the Wess–Zumino condition corresponds to classifying local cohomology classes of the nilpotent operator $\hat{\delta}=(\cdot,S)$ (1.4) at ghost number one. At higher order this same classification will also be involved in characterising a possible arbitrariness in the solution of (4.10), but also a particular solution must be found. The algebraic analysis of the higher loop relations (4.10) and the characterization of higher loop BRST anomalies falls however outside the scope of this paper.

5 The Two–loop Anomaly in Chiral W_3 gravity

In the previous sections we have deduced the form of genuine, local higher loop anomalies and of their consistency conditions from the quantum master equation (1.7). Now we proceed to an illustration, and an explicit verification of the resulting expressions (3.13) and (4.9), in the example of chiral W_3 gravity –for which the corresponding BRST anomaly gets contributions up to two loops.

5.1 Nonlocally Regulated Chiral W_3 Gravity

We use the conventions and notations of refs. [25] and [20]. This subsection contains selected material from these papers, to which we refer the reader for the actual construction of the proper solution of the master equation and the nonlocalization process for W_3 gravity respectively.

Chiral W_3 gravity [26] can be realised as a system of D two dimensional scalar fields ϕ^i, $i = 1, \ldots, D$, coupled to gauge fields h and B through the chiral spin-2 and spin-3 currents

$$T = \frac{1}{2}(\partial\phi^i)(\partial\phi^i), \qquad W = \frac{1}{3}d_{ijk}(\partial\phi^i)(\partial\phi^j)(\partial\phi^k),$$

defined in terms of the usual combinations of space-time derivatives $\partial = \partial_+$, $\bar\partial = \partial_-$, with $x^\pm = \frac{1}{\sqrt{2}}(x^1 \pm x^0)$, and of a constant, totally symmetric tensor d_{ijk} satisfying the identity

$$d_{i(jk}d_{l)mi} = k\delta_{(jl}\delta_{k)m},$$

for an arbitrary, but fixed parameter k.

The proper solution of the classical master equation for this system [25] is

$$\begin{aligned} S = \int \mathrm{d}^2x \quad & \left\{ \left[-\frac{1}{2}(\partial\phi^i)(\bar\partial\phi^i) + b(\bar\partial c) + v(\bar\partial u) \right] \right. \\ & +\phi^*_i \left[c(\partial\phi^i) + u d_{ijk}(\partial\phi^j)(\partial\phi^k) - 2kb(\partial u)u(\partial\phi^i) \right] \\ & +b^* \left[-T + 2b(\partial c) + (\partial b)c + 3v(\partial u) + 2(\partial v)u \right] \\ & +v^* \left[-W + 2kTb(\partial u) + 2k\partial(Tbu) + 3v(\partial c) + (\partial v)c \right] \\ & \left. +c^* \left[(\partial c)c + 2kT(\partial u)u \right] + u^* \left[2(\partial c)u - c(\partial u) \right] \right\} \\ = \quad & \mathcal{S}(\Phi) + \Phi^*_A R^A(\Phi), \end{aligned} \tag{5.1}$$

where $\{c, u\}$ stand for the ghosts corresponding to spin-2 and spin-3 gauge symmetries; $\{b, v\}$, for their associated antighosts; and $\{\phi^*_i, c^*, u^*, b^*, v^*\}$, for the corresponding antifields. This gauge–fixed action is obtained by performing a canonical transformation from the classical basis of fields and antifields $\{\phi^i, h, B, c, u; \phi^*_i, h^*, B^*, c^*, u^*\}$ to the so-called gauge–fixed basis:

$$\{h, h^*, B, B^*\} \to \{b = h^*, b^* = -h, v = B^*, v^* = -B\}.$$

In this way, the classical interaction is completely contained in the antifield dependent part, $\Phi^*_A R^A(\Phi)$, so that antifields play the role of coupling constants.

Nonlocal regularization of the quantum theory stemming from the proper solution (5.1) is described in [20]. The propagating fields are ordered in a vector $\Phi^A = \{\phi^i; b, v; c, u\}$. The kinetic operator F_{AB} in (2.1) is

$$F_{AB} = \begin{pmatrix} \partial\bar\partial\,\delta_{ij} & 0 & 0 \\ 0 & 0 & \mathbf{1}\,\bar\partial \\ 0 & \mathbf{1}\,\bar\partial & 0 \end{pmatrix},$$

where $\mathbf{1}$ stands for the identity in the spin 2 (spin 3) ghost sector. The operator $(T^{-1})^{AB}$ is chosen as

$$(T^{-1})^{AB} = \begin{pmatrix} \delta^{ij} & 0 & 0 \\ 0 & 0 & \mathbf{1}\,\partial \\ 0 & \mathbf{1}\,\partial & 0 \end{pmatrix},$$

and yields a simple regulator $\mathcal{R}^A_B$ (2.2) and smearing operator ε^A_B (2.3):

$$\mathcal{R}^A_B = \partial\bar{\partial}\,\delta^A_B, \qquad \varepsilon^A_B = \exp\left(\frac{\partial\bar{\partial}}{2\Lambda^2}\right)\delta^A_B \equiv \varepsilon\,\delta^A_B.$$

Finally, the shadow propagator $\mathcal{O}^{AB}$ (2.4) is

$$\mathcal{O}^{AB} = \begin{pmatrix} \mathcal{O} & 0 & 0 \\ 0 & 0 & \mathbf{1}\mathcal{O}\partial \\ 0 & \mathbf{1}\mathcal{O}\partial & 0 \end{pmatrix}, \qquad \text{with} \qquad \mathcal{O} \equiv \frac{(\varepsilon^2-1)}{\partial\bar{\partial}} = \int_0^1 \frac{\mathrm{d}t}{\Lambda^2}\exp\left(t\frac{\partial\bar{\partial}}{\Lambda^2}\right). \quad (5.2)$$

To complete the specification of the regulated theory we have to provide the counterterms M_p. These are generically necessary to ensure finiteness of the theory. However, chiral W_3 gravity is "finite", as is well known, so that here inclusion of such counterterms is in principle not necessary. One might still want to include finite counterterms (to preserve BRST invariance as far as possible), but we will not do this here. Then W is just the same as the classical action S (5.1), and algebraic computations involving the operator Δ simplify accordingly. In the present case, we have for S^A_B

$$S^A_B = \begin{pmatrix} c^i_j\partial & -2k(\partial u)u(\partial\phi^i) & 0 & (\partial\phi^i) & u^i \\ -(\partial\phi_j)\partial & -(c\partial)_2 & -2(u\partial)_{3/2} & (b\partial)_1 & 3(v\partial)_{1/3} \\ -u_j\partial & -2k[T(u\partial)_2+u(\partial T)] & -(c\partial)_3 & 3(v\partial)_{1/3} & 4k(bT\partial)_{1/2} \\ 2k(\partial u)u(\partial\phi_j)\partial & 0 & 0 & -(c\partial)_{-1} & -2kT(u\partial)_{-1} \\ 0 & 0 & 0 & -2(u\partial)_{-1/2} & -(c\partial)_{-2} \end{pmatrix}, \quad (5.3)$$

with c^i_j and u^i the operators

$$\begin{aligned} c^i_j &= \left[c\delta^i_j - 2kb(\partial u)u\delta^i_j + 2ud^i_{jk}(\partial\phi^k)\right], \quad (5.4) \\ u^i &= d^i_{jk}(\partial\phi^j)(\partial\phi^k) - 2k\left[b(\partial u)(\partial\phi^i) + (b(\partial\phi^i)u\partial)_1\right], \end{aligned}$$

and where $(F(\Phi,\Phi^*)\,\partial)_n$ stands for the shorthand notation

$$(F\,\partial)_n = F\,\partial + n(\partial F), \qquad (F\,\partial)^\dagger_n = -[F\,\partial + (1-n)(\partial F)] = -(F\,\partial)_{1-n}.$$

In much the same way, the operator $(I_{\mathrm{cl}})_{AB}$ reads

$$(I_{\mathrm{cl}})_{AB} = \begin{pmatrix} \partial\,h^*_{ij}\,\partial & (g^*_i\partial)_1 & 0 & -(\phi^*_i\partial)_1 & (q^*_i\partial)_1 \\ g^*_j\,\partial & 0 & 0 & (b^*\partial)_{-1} & r^* \\ 0 & 0 & 0 & 2(v^*\partial)_{-1/2} & (b^*\partial)_{-2} \\ -\phi^*_j\,\partial & (b^*\partial)_2 & 2(v^*\partial)_{3/2} & 2(c^*\partial)_{1/2} & -3(u^*\partial)_{2/3} \\ q^*_j\,\partial & -(r^*)^\dagger & (b^*\partial)_3 & -3(u^*\partial)_{1/3} & 2(p^*\partial)_{1/2} \end{pmatrix}. \quad (5.5)$$

The following abbreviations have been used for the matrix elements, all linear quantities in the antifields:

$$\begin{aligned} h^*_{ij} &= \delta_{ij}\left[b^* + 2kb(u(\partial v^*) - v^*(\partial u)) + 2kc^*u(\partial u)\right] - 2d_{ij}{}^k\phi^*_k u + 2v^*d_{ijk}(\partial\phi^k), \quad (5.6) \\ g^*_i &= 2k\left[\phi^*_i(\partial u)u + (v^*(\partial u) - u(\partial v^*))(\partial\phi_i)\right], \\ q^*_i &= -2\phi^*_j d^j_{ik}(\partial\phi^k) + 2k\left[(\partial(v^*b(\partial\phi_i) + u\phi^*_i b)) + b(\partial\phi_j)(v^*\partial)_1 + \phi^*_i b(u\partial)_1\right], \\ r^* &= 2k\left[T(v^*\partial)_{-1} - \phi^*_i(\partial\phi^i)(u\partial)_{-1}\right], \\ p^* &= 2k\left[Tc^* - \phi^*_i(\partial\phi^i)b\right]. \end{aligned}$$

In the absence of counterterms, the nonlocally regulated anomaly $\mathcal{A}_{\Lambda R}$ (2.17) simplifies to the regulated trace (3.2):

$$\mathcal{A}_{\Lambda R}(\Phi,\Phi^*) = \Delta S_\Lambda(\Phi,\Phi^*) = \Omega_S(\Phi+\bar{\Psi}_0, \Phi^*\varepsilon^2). \tag{5.7}$$

According to eq. (3.4), all the information about the anomalies in this theory is contained in this quantity and its quantum corrections:

$$\begin{aligned} \lim_{\Lambda^2\to\infty}(\Delta S_\Lambda\cdot\Gamma_\Lambda) &= \lim_{\Lambda^2\to\infty}\left[(\Delta S_\Lambda)+\hbar(Q_1\Delta S_\Lambda)+O(\hbar^2)\right] \\ &= \mathcal{A}_1+\hbar\left[\mathcal{A}_2+(\mathcal{A}_1\cdot\Gamma_1)\right]+O(\hbar^2). \end{aligned} \tag{5.8}$$

We now disentangle from this expression the one-loop anomaly (a computation performed in [20]), and then the two–loop anomaly.

5.2 One–loop anomaly

The relevant local contributions in the loop expansion (5.8) can be inferred from considerations of dimensions and spin: they have the general form

$$\Lambda^{-2(m-n)}(\Phi^*)^{m+1}\bar{\partial}^{-n}F_{m,n}(\Phi;\partial), \qquad m,n=0,1,\ldots, \qquad m\geq n.$$

By locality, $n=0$ for the $\mathcal{A}_i$ terms: intrinsic nonlocalities of the form $(\bar{\partial})^{-n}$, $n>0$ are associated with quantum dressings of such local anomalies. Also, in the limit $\Lambda\to\infty$ only terms with $m=0$ remain, which are therefore linear in the antifields. Evaluating these linear terms proceeds by expansion of Ω_S (5.7), given in terms of S^A_B, (5.3), and $(I_{\rm cl})_{AB}$, (5.5):

$$\Omega^{(1)}_S = \left[\varepsilon^2\, S^A_B\,\mathcal{O}^{BC}\,(I_{\rm cl})_{CA}\right], \qquad\qquad \Omega_S=\Omega^{(1)}_S+O((\Phi^*)^2). \tag{5.9}$$

Separating the terms corresponding to the different antifields, upon taking the limit $\Lambda^2\to\infty$, five different contributions are obtained for the final expression for the one–loop anomaly:

$$\mathcal{A}_1 = \lim_{\Lambda^2\to\infty}\Omega^{(1)}_S = \mathcal{A}^{(i)}_1+\mathcal{A}^{(ii)}_1+\mathcal{A}^{(iii)}_1+\mathcal{A}^{(iv)}_1+\mathcal{A}^{(v)}_1. \tag{5.10}$$

The different contributions are given by [20]

$$\begin{aligned} \mathcal{A}^{(i)}_1 &= \frac{i}{24\pi}\int \mathrm{d}^2x\, c^{ij}\,\partial^3\, h^*_{ij}, \\ \mathcal{A}^{(ii)}_1 &= \frac{-100\,i}{24\pi}\int \mathrm{d}^2x\, c\,\partial^3\, b^*, \\ \mathcal{A}^{(iii)}_1 &= \frac{ik}{2\pi}\int \mathrm{d}^2x\,(v^*(\partial u)-u(\partial v^*))(\partial\phi^i)(\partial^3\phi^i), \\ \mathcal{A}^{(iv)}_1 &= \frac{ik}{6\pi}\int \mathrm{d}^2x\, T\left[5(\partial^3u)v^*-12(\partial^2u)(\partial v^*)+12(\partial u)(\partial^2v^*)-5u(\partial^3v^*)\right], \\ \mathcal{A}^{(v)}_1 &= \frac{-ik}{6\pi}\int \mathrm{d}^2x\,\phi^*_i\left[6\partial\left(u(\partial u)(\partial^2\phi^i)\right)+9(\partial^2u)(\partial u)(\partial\phi^i)+8u(\partial^3u)(\partial\phi^i)\right], \end{aligned} \tag{5.11}$$

and are in complete agreement with previous computations in the literature, using either PV regularization [25] or standard conformal field theory techniques [27].

For the computation of the two–loop anomaly however, the limiting expression for the one–loop anomaly is not sufficient: as emphasized in the general section, we need

information that is only contained in the expression for the regulated anomaly *before* the limit. The corresponding expressions for the different parts in $\Omega_S^{(1)}$,

$$\Omega_S^{(1)} = \Omega_S^{(1,i)} + \Omega_S^{(1,ii)} + \Omega_S^{(1,iii)} + \Omega_S^{(1,iv)} + \Omega_S^{(1,v)},$$

were also obtained in the course of the computations of [20], but not recorded there, so we give them now:

$$\begin{aligned}
\Omega_S^{(1,i)} &= \frac{-i}{2\pi}\int \mathrm{d}^2x\, c^{ij}\,\bar{\mathcal{O}}(1,1;\Lambda^2)\,\partial^3\, h^*_{ij}, \qquad (5.12)\\
\Omega_S^{(1,ii)} &= \frac{i}{2\pi}\int \mathrm{d}^2x\, c\,[4\bar{\mathcal{O}}(1,1;\Lambda^2) - 16\bar{\mathcal{O}}(0,0;\Lambda^2)]\,\partial^3\, b^*,\\
\Omega_S^{(1,iii)} &= \frac{ik}{\pi}\int \mathrm{d}^2x\,(\partial^3\phi^i)\bar{\mathcal{O}}(0,0,\Lambda^2)\ \left[(v^*(\partial u) - u(\partial v^*))(\partial\phi^i)\right],\\
\Omega_S^{(1,iv)} &= \frac{-ik}{\pi}\int \mathrm{d}^2x\, T\ \left\{v^*\left[4\bar{\mathcal{O}}(1,1;\Lambda^2) - \bar{\mathcal{O}}(0,0;\Lambda^2)\right](\partial^3 u)\right.\\
&\qquad \left. +4(\partial v^*)\bar{\mathcal{O}}(0,0;\Lambda^2)(\partial^2 u) - (u\leftrightarrow v^*)\right\},\\
\Omega_S^{(1,v)} &= \frac{ik}{\pi}\int \mathrm{d}^2x\,\phi^*_i\left\{(\partial u)\,u\,\bar{\mathcal{O}}(0,0;\Lambda^2)\,(\partial^3\phi^i) + \bar{\mathcal{O}}(0,0;\Lambda^2)\,\partial^2\left((\partial u)u(\partial\phi^i)\right)\right.\\
&\quad \left. +4(\partial\phi^i)(\partial u)\bar{\mathcal{O}}(0,0;\Lambda^2)(\partial^2 u) - 4(\partial\phi^i)u\left[\bar{\mathcal{O}}(0,0;\Lambda^2) + \bar{\mathcal{O}}(0,1;\Lambda^2)\right](\partial^3 u)\right\},
\end{aligned}$$

where again the abbreviations c^i_j (5.4), h^*_{ij} (5.6) were used, and the operator $\bar{\mathcal{O}}(n,m;\Lambda^2)$ is defined as

$$\bar{\mathcal{O}}(n,m;\Lambda^2) = \int_0^1 \mathrm{d}t\,\frac{(-1)^m\, t^m}{(1+t)^{n+m+2}}\exp\left(\frac{t}{1+t}\,\frac{\partial\bar{\partial}}{\Lambda^2}\right). \qquad (5.13)$$

In the limit, the operators $\bar{\mathcal{O}}(n,m;\Lambda^2)$ in these expressions become numerical coefficients, resulting in (5.11).

5.3 Two–loop anomaly

The evaluation of the two–loop anomaly is greatly facilitated by the fact that, as noted in the previous subsection, it depends linearly on the antifields. Since $\Omega_S^{(1)}$ is already linear, see (5.9), one may neglect the antifield dependence in the rest of (3.12)

$$(S^{-1})^{AB} = (F^{-1})^{AB} + O(\Phi^*), \qquad (\delta_\Lambda^{(0)})^A_{\ B} = \delta^A_B + O(\Phi^*).$$

The local part that eventually may arise from the one–loop dressing $(Q_1\Delta S_\Lambda)$ of the insertion (5.7) is then

$$Q_1\,\Delta S_\Lambda = \frac{i}{2}\left[\varepsilon^2(F^{-1})^{AB}\left(\Omega_S^{(1)}\right)_{BA}\right](\Phi + \bar{\Psi}_0, \Phi^*\varepsilon^2) + O((\Phi^*)^2).$$

We require the full Λ dependence of $\Omega_S^{(1)}$ (5.9), which was given in (5.12). Similar arguments determine the possible local contributions generated by the genuine, local one-loop anomaly $\mathcal{A}_1$ (5.10) to be of an analogous form

$$Q_1\mathcal{A}_1(\Phi + \bar{\Psi}_0, \Phi^*\varepsilon^2) = \frac{i}{2}\left[\varepsilon^2(F^{-1})^{AB}(\mathcal{A}_1)_{BA}\right](\Phi + \bar{\Psi}_0, \Phi^*\varepsilon^2) + O((\Phi^*)^2).$$

The two–loop anomaly resulting from this analysis of (3.13), and in the absence of counterterms, is given by

$$\mathcal{A}_2 = \lim_{\Lambda^2\to\infty}\left\{\frac{i}{2}\left[\varepsilon^2(F^{-1})^{AB}\left(\Omega_S^{(1)} - \mathcal{A}_1\right)_{BA}\right]\right\}. \qquad (5.14)$$

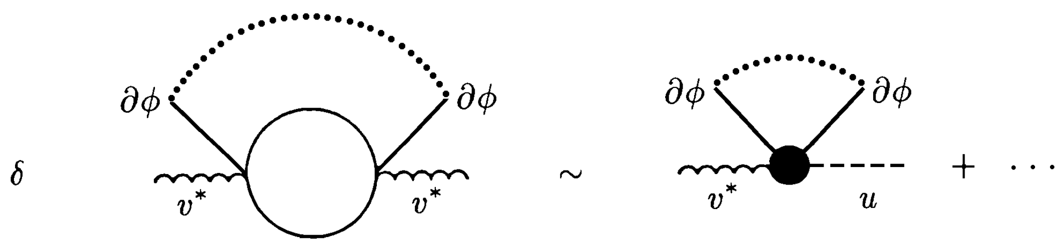

Figure 1: The diagram on the left (without the dotted line) represents a contribution to Γ_1 with two external matter lines. Its variation δ yields (a.o.) the one–loop anomaly represented by the diagram on the right (again without the dotted line). Contracting the external matter lines, the diagrams contribute to Γ_2 and the two–loop anomaly respectively. In that case it is essential that the vertex on the r.h.s. is nonlocalised: the fat blob stands for the smearing operator $\bar{\mathcal{O}}$ of eq. (5.13).

The interpretation of this formula is that the two–loop anomaly results by contracting external lines (the factor F^{-1}) in a regularised fashion (the factor ε^2). The expression that undergoes this contraction is however not the one–loop anomaly (which is $\mathcal{A}_1$), but rather the (evanescent) difference between the *regulated* one-loop anomaly and the one–loop anomaly itself.

The two loop anomaly can be computed from (5.14) in a straightforward manner. We will, for illustrative purposes, pick out a specific term and show in detail the mechanism to generate $\mathcal{A}_2$. Afterwards, we will give the formulas needed for the complete calculation, with considerably less detail.

A specific non–zero contribution to the two–loop anomaly arises from the last term of h^*_{ij} in (5.6) via $\Omega_S^{(1,i)}$ of (5.12) with the last term in c_{ij}, see (5.4). The corresponding diagrams are given in figure 1. At one loop, in Γ, the diagram with two external ϕ lines and two factors v^* gives rise to an anomalous variation, in (Γ, Γ), which contains (among others) the term under consideration. At the two–loop level, a contribution arising from "closing the ϕ propagator" can similarly give rise to a two loop anomaly. In the figure, the smeared out vertex symbolizes the fact that not just the local $\Lambda \to \infty$ part is to be taken (in which case the blob would be a point), but one should keep non–leading parts also. The corresponding double momentum integral for the exact expression, in non–local regularisation, takes the form

$$\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \int_0^1 \mathrm{d}t \, \frac{t}{(t+1)^4} \, \frac{(p+k)^3 \, k^2}{k \, \bar{k}} \, \exp\left(-\frac{t}{t+1} \frac{|p+k|^2}{2\Lambda^2}\right) \exp\left(-\frac{|k|^2}{2\Lambda^2}\right).$$

One recognises the three momentum factors corresponding to the three derivatives on h^*_{ij} in $\Omega_S^{(1,i)}$, the two momentum factors on the contracted ϕ-line, and the scalar field propagator. There are two cutoff factors, one corresponding to the explicit ε^2 in formula (5.14) and the other to the Λ dependence of $\Omega_S^{(1,i)}$ as given by the factor $\bar{\mathcal{O}}(1,1;\Lambda^2)$. The second term in (5.14), corresponding to removing the one–loop dressing of the one–loop anomaly, has exactly the same form but without this last factor, i.e. $\bar{\mathcal{O}}(1,1;\Lambda^2) \to \bar{\mathcal{O}}(1,1;\infty) = -1/12$. The integral is well–defined in both cases. In the latter case, the angular k-integral shows that it vanishes. In the former case, the integral is equal to

$$\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \int_0^1 \mathrm{d}t \, \frac{t}{(t+1)^4} \, \frac{(p+k)^3 \, k^2}{k \, \bar{k}} \, \exp\left(-\frac{2t+1}{t+1} \frac{|k|^2}{2\Lambda^2}\right) \exp\left(-\frac{t}{t+1} \frac{|p|^2 + p\bar{k} + \bar{p}k}{2\Lambda^2}\right). \tag{5.15}$$

A variety of methods exist to compute the $\Lambda \to \infty$ limit of this integral. It is instructive to evaluate it by a power series expansion of the last factor in (5.15). The first few terms are again zero due to the angular integration. In fact, only terms with $\bar{k}^n$ with $n = 2, 3, 4, 5$, give nonvanishing contributions. The integral then becomes, in the limit, equal to a numerical factor times p^5.

Several aspects of this sample calculation merit some comment. First, it is seen quite explicitly that taking the $\Lambda \to \infty$ limit too early, i.e. replacing $\bar{\mathcal{O}}(1,1;\Lambda^2)$ in $\Omega_S^{(1,i)}$ by its limiting value, would be a mistake and give a vanishing overall result. From the reasoning above it follows that dressing of terms with inverse powers of Λ^2 result in finite contributions. This is the evanescent aspect of the difference. A second comment is that the intuitive idea, that the two–loop anomaly results from a ϕ-dependent one–loop anomaly by closing the ϕ–loop, has some truth but is certainly not the complete truth. Indeed, closing this loop on the final one–loop anomaly in fact gives a *vanishing result* (recall the discussion of the second term of (5.14) in the instance above). It does arise from such a diagram however if one takes the one–loop anomaly in its regulated form *before taking the limit*: the non–zero contribution is due to terms in the one–loop anomaly that have inverse powers of the cutoff Λ. A third comment is that the power series expansion as performed on the integral above automatically yields a polynomial in the momentum p. This attests the fact that the resulting expression is local.

Having illustrated the mechanism by considering one term in detail, we now give a more complete account. Starting again from (5.14), due to the simple form of the free propagator $(F^{-1})^{AB}$ only a few of the matrix elements of $\Omega_S^{(1)}$ are involved, namely

$$
\begin{aligned}
\left(\Omega_S^{(1)}\right)_{ij} &= \frac{2i}{\pi} d_{ij} \left[\partial\, u\, \bar{\mathcal{O}}(1,1;\Lambda^2)\partial^3\, v^*\, \partial - \partial\, v^* \bar{\mathcal{O}}(1,1;\Lambda^2)\partial^3\, u\, \partial\right] \\
&\quad + \frac{ik}{\pi}\delta_{ij}\left[\bar{\mathcal{O}}(0,0;\Lambda^2)\partial^3(u(\partial v^*) - v^*(\partial u))\partial + \partial(u(\partial v^*) - v^*(\partial u))\partial^3\bar{\mathcal{O}}(0,0;\Lambda^2)\right] \\
&\quad + \frac{ik}{\pi}\delta_{ij}\left\{\partial\left[v^*\left(4\bar{\mathcal{O}}(1,1;\Lambda^2) - \bar{\mathcal{O}}(0,0;\Lambda^2)\right)(\partial^3 u)\right]\partial \right.\\
&\qquad\qquad \left. + 4\partial\left[(\partial v^*)(\bar{\mathcal{O}}(0,0;\Lambda^2)\partial^2 u)\partial\right] - (u \leftrightarrow v^*)\right\}, \\
\left(\Omega_S^{(1)}\right)_{bc} &= \frac{iDk}{\pi}(u(\partial v^*) - v^*(\partial u))\bar{\mathcal{O}}(1,1;\Lambda^2)\partial^3, \qquad \left(\Omega_S^{(1)}\right)_{cb} = -\left(\Omega_S^{(1)}\right)^\dagger_{bc},
\end{aligned}
$$

(with $d_{ij} \equiv d_{ilm}d_j^{\ lm}$) and their local counterparts $(\mathcal{A}_1)_{AB}$ obtained by taking the limit $\Lambda^2 \to \infty$ (some of these are given later, in eq. (5.21)). The expression (5.14) reduces to the functional trace

$$
\mathcal{A}_2(u,v^*) = \lim_{\Lambda^2\to\infty}\left\{\frac{i}{2}\,\mathrm{tr}\left[\varepsilon^2\left(\frac{\delta^{ij}}{\partial\bar{\partial}}\left(\Omega_S^{(1)} - \mathcal{A}_1\right)_{ji} + \frac{1}{\bar{\partial}}\left(\Omega_S^{(1)} - \mathcal{A}_1\right)_{bc} + \left(\Omega_S^{(1)} - \mathcal{A}_1\right)_{bc}\frac{1}{\bar{\partial}}\right)\right]\right\}, \tag{5.16}
$$

in which only a functional dependence on the fields u and v^* appears.

Two types of terms are present, proportional to one of the two parameter combinations $d^2 = d_{ijk}d^{ijk}$ and kD that arise upon taking traces with respect the discrete matter indices. Terms proportional to kD are always of the form

$$
\mathrm{tr}\left[F(x)\bar{\mathcal{O}}(n,m;\Lambda^2)\frac{\partial^3}{\bar{\partial}}\varepsilon^2\right] \quad \text{or} \quad \mathrm{tr}\left[G(x)\frac{\partial^3}{\bar{\partial}}\varepsilon^2\right],
$$

where $F(x)$ and $G(x)$ are *functions* of the fields u and v^*. The corresponding traces vanish, as can be seen by writing them out in momentum space and doing the angular integration, along the same lines as in the sample calculation above. The same result holds true for

all the terms (including the d^2 contributions) generated by the one–loop anomaly $\mathcal{A}_1$. A nonvanishing result for the traces we are dealing with may only result by the presence of $\bar{\mathcal{O}}(n,m;\Lambda^2)$ operators (5.13) *between* the fields u and v^*. This happens for the remaining d^2 contributions which read, after trivial manipulations,

$$\frac{-2\hbar d^2}{\pi}\,\mathrm{tr}\left[u\,\bar{\mathcal{O}}(1,1;\Lambda^2)\,\partial^3\,v^*\,\frac{\partial}{\bar{\partial}}\,\varepsilon^2\right]. \tag{5.17}$$

Writing out this functional trace in momentum space yields

$$\frac{-2i\hbar d^2}{\pi}\int \mathrm{d}^2x\,u(x)\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,v^*(p)\int_0^1\mathrm{d}t\,\frac{t}{(1+t)^4}$$
$$\times\left\{\int\frac{\mathrm{d}^2k}{(2\pi)^2}\,\exp\left(-\frac{t}{1+t}\,\frac{(p+k)(\bar{p}+\bar{k})}{\Lambda^2}-\frac{k\bar{k}}{\Lambda^2}\right)\,\frac{(p+k)^3k}{\bar{k}}\right\}.$$

After shifting $k^\mu\;\rightarrow\;\left(\sqrt{\frac{t+1}{2t+1}}\,k^\mu-\frac{t}{2t+1}\,p^\mu\right)$ to decouple the integrals over k and p we get

$$\frac{-2i\hbar d^2}{\pi}\int\mathrm{d}^2x\,u(x)\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,v^*(p)\int_0^1\mathrm{d}t\,\frac{t}{(1+t)^3(2t+1)}\,\exp\left(-\frac{t}{2t+1}\,\frac{p\bar{p}}{\Lambda^2}\right)\mathcal{I}(t,p;\Lambda^2),$$

in terms of the momentum integral $\mathcal{I}(t,p;\Lambda^2)$

$$\mathcal{I}(t,p;\Lambda^2)=\int\frac{\mathrm{d}^2k}{(2\pi)^2}\,\exp\left(-\frac{k\bar{k}}{\Lambda^2}\right)\frac{\left(\sqrt{\frac{t+1}{2t+1}}\,k-\frac{t}{2t+1}\,p\right)\left(\sqrt{\frac{t+1}{2t+1}}\,k+\frac{t+1}{2t+1}\,p\right)^3}{\left(\sqrt{\frac{t+1}{2t+1}}\,\bar{k}-\frac{t}{2t+1}\,\bar{p}\right)}. \tag{5.18}$$

The net result of the presence of the operator $\bar{\mathcal{O}}(1,1;\Lambda^2)$ in (5.17) is the p–dependent shift of the integrand in (5.18), without which the expression would vanish. The angular integral can easily be done by converting it into a complex contour integral, using $k=i\sqrt{\frac{x}{2}}\,|p|\,z$ and $\bar{k}=-i\sqrt{\frac{x}{2}}\,|p|\,\frac{1}{z}$:

$$\begin{aligned}\mathcal{I}(t,p;\Lambda^2)\;&=\;-\frac{p^5}{(2\pi)^2}\,\frac{2t+1}{t}\int_0^\infty\mathrm{d}x\,\exp\left(-\frac{x}{2}\,\frac{|p|^2}{\Lambda^2}\right)\\&\quad\times\oint_{S_1}\mathrm{d}z\frac{\left(\sqrt{x}z\sqrt{\frac{t+1}{2t+1}}+\frac{t+1}{2t+1}\right)\left(\sqrt{x}z\sqrt{\frac{t+1}{2t+1}}-\frac{t}{2t+1}\right)}{\left(z-\sqrt{x}\,\frac{\sqrt{(t+1)(2t+1)}}{t}\right)},\end{aligned}$$

where the contour integral in the complex plane is counterclock-wise over the unit circle. There is just one single pole, and only for a limited range of values of x, which cuts off the x-integral to $x\leq x_0(t)\equiv\frac{t^2}{(t+1)(2t+1)}$. In terms of the rescaled variable $y=x/x_0(t)$ one finds

$$\mathcal{I}(t,p;\Lambda^2)=\frac{-(ip)^5}{2\pi}\,\frac{t^5}{(t+1)(2t+1)^4}\int_0^1\mathrm{d}y\,\left(y+\frac{t+1}{t}\right)^3(y-1)\,\exp\left(-x_0(t)\,y\,\frac{p\bar{p}}{\Lambda^2}\right).$$

At last the $\Lambda\rightarrow\infty$ limit can be taken. The unique nonvanishing contribution to our expression (5.16) for the two–loop anomaly becomes

$$\lim_{\Lambda^2\to\infty}\left\{\frac{-2\hbar d^2}{\pi}\,\mathrm{tr}\left[u\,\bar{\mathcal{O}}(1,1;\Lambda^2)\,\partial^3\,v^*\,\frac{\partial}{\bar{\partial}}\varepsilon^2\right]\right\}=\frac{i\hbar d^2}{\pi^2}\,a\int\mathrm{d}^2x\,u\,\partial^5\,v^*,$$

with the numerical factor a given by the integral

$$a = \int_0^1 \mathrm{d}t \, \frac{t^6}{(t+1)^4(2t+1)^5} \int_0^1 \mathrm{d}y \, \left(y + \frac{t+1}{t} \right)^3 (y-1) = \frac{1}{6!}.$$

In summary, the expression for the two–loop anomaly (5.16) arising out of this computation is

$$\mathcal{A}_2(v^*, u) = \frac{i\hbar d^2}{720\pi^2} \int \mathrm{d}^2 x \, u \, \partial^5 \, v^*. \tag{5.19}$$

This agrees with previous results in the literature, obtained from operator product expansions [27], or from the nonlocally regulated two-loop effective action [20].

5.4 Two–loop Consistency Condition

With the explicit expressions (5.10) and (5.19) of the one and two loop anomalies for chiral W_3 gravity at hand, we can finally pass to the verification of the two-loop consistency condition (4.9). The BRST variation of the two–loop anomaly is

$$\begin{aligned}(\mathcal{A}_2, S) \quad &= \quad \frac{id^2}{720\pi^2} \int \mathrm{d}^2 x \left\{ (\partial^5 v^*)[2(\partial c)u - c(\partial u)] + (\partial^5 u)[2(\partial c)v^* - c(\partial v^*)] \right\} \\ &\quad + \frac{id^2}{720\pi^2} \int \mathrm{d}^2 x (\partial^5 u)[b^*(\partial u) - 2(\partial b^*)u]. \end{aligned} \tag{5.20}$$

The fact that it does not vanish confirms, notwithstanding ref. [16], that higher loop consistency conditions in general do not have the one–loop form (1.12), but require the presence of extra pieces related with the action of Δ on lower order anomalies. In the absence of counterterms M_p, the computation of such extra terms using our general result eqs. (4.8)–(4.9), requires knowledge of the matrix of second derivatives of the one–loop anomaly (5.11) with respect to fields and antifields. The relevant nonvanishing entries of $(\mathcal{A}_1)_{AB}$ are:

$$\begin{aligned}(\mathcal{A}_1)_{ij} \quad &= \quad -\frac{i}{6\pi} d_{ij} (\partial \, u \partial^3 \, v^* \, \partial - \partial \, v^* \, \partial^3 \, u \, \partial) \\ &\quad + \frac{ik}{2\pi} \delta_{ij} \left[\partial(u(\partial v^*) - v^*(\partial u))\partial^3 + \partial^3(u(\partial v^*) - v^*(\partial u))\partial \right] \\ &\quad - \frac{ik}{6\pi} \delta_{ij} \partial \left[5(\partial^3 u)v^* - 12(\partial^2 u)(\partial v^*) + 12(\partial u)(\partial^2 v^*) - 5u(\partial^3 v^*) \right] \partial, \\ (\mathcal{A}_1)_{ib} \quad &= \quad \frac{ik}{6\pi} d_i \left[\partial \, u \, \partial^3 \, (u(\partial v^*) - v^*(\partial u)) - \partial \, v^* \, \partial^3 (\partial u) u \right], \\ (\mathcal{A}_1)_{ic} \quad &= \quad \frac{i}{12\pi} d_i \partial \, v^* \, \partial^3, \qquad (\mathcal{A}_1)_{bc} = \frac{-iDk}{12\pi} (u(\partial v^*) - v^*(\partial u))\partial^3, \\ (\mathcal{A}_1)_{bu} \quad &= \quad \frac{ik}{12\pi} \left[\partial^3(Db^* - 2\phi^* u + 2v^*(\partial\phi))(u\partial)_{-1} - \partial^3(Dc + 2u(\partial\phi))(v^*\partial)_{-1} \right. \\ &\qquad \left. + 2(\partial u) u \, \partial^3 \, \phi^* - 2(u(\partial v^*) - v^*(\partial u))\partial^3(\partial\phi) \right], \end{aligned} \tag{5.21}$$

where d_i stands for the contraction $d_i = \delta^{jk} d_{ijk}$ and ϕ, ϕ^* for the field and antifield combinations $\phi^i d_i$, $\phi^*_i d^i$, respectively. On the other hand, the nonvanishing relevant entries for $(\mathcal{A}_1)^A_B$ are:

$$\begin{aligned}(\mathcal{A}_1)^i_{\ j} \quad &= \quad -\frac{ik}{6\pi} \delta^i_j \left[6\partial \, u(\partial u)\partial^2 + 9(\partial^2 u)(\partial u)\partial + 8u(\partial^3 u)\partial \right] + \frac{i}{6\pi} d^i_j u \, \partial^3 \, u \, \partial, \\ (\mathcal{A}_1)^i_{\ b} \quad &= \quad -\frac{ik}{6\pi} d^i u \, \partial^3 (\partial u) u, \qquad (\mathcal{A}_1)^i_{\ c} = \frac{i}{12\pi} d^i \, u \, \partial^3, \end{aligned}$$

$$
\begin{aligned}
(\mathcal{A}_1)_i^{b^*} &= -\frac{i}{12\pi} d_i\,\partial^3\,u\,\partial, \qquad (\mathcal{A}_1)_b^{b^*} = \frac{ikD}{12\pi}\partial^3(\partial u)u, \\
(\mathcal{A}_1)_b^{v^*} &= \frac{ik}{12\pi}\left[(\partial\phi)\partial^3(\partial u)u + 2(\partial u)\partial^3(Dc + 2u(\partial\phi)) + 2\partial\,u(\partial^3(Dc+2u(\partial\phi)))\right], \\
(\mathcal{A}_1)_i^{c^*} &= -\frac{ik}{6\pi} d_i u(\partial u)\partial^3\,u\,\partial, \qquad (\mathcal{A}_1)_c^{c^*} = -\frac{ikD}{12\pi}u(\partial u)\partial^3, \\
(\mathcal{A}_1)_u^{c^*} &= -\frac{ik}{12\pi}\left[D(\partial^3 c)(u\partial)_{-1} + 2u(\partial u)\partial^3(\partial\phi) + 2\partial^3\,u(\partial\phi)(u\partial)_{-1}\right].
\end{aligned}
$$

The further computation of $\Omega_{\mathcal{A}_1}$ (4.8) is again organized by means of an expansion in the number of antifields

$$
\Omega_{\mathcal{A}_1} = (-1)^{(A+1)}\left\{\left[\varepsilon^2\,(\mathcal{A}_1)_A^A\right] + \left[\varepsilon^2\,\left((\mathcal{A}_1)_B^A\mathcal{O}^{BC}\,I_{CA} + S_B^A\mathcal{O}^{BC}(\mathcal{A}_1)_{CA}\right)\right]\right\} + O((\Phi^*)^2), \tag{5.22}
$$

and as before only the finite terms linear in antifields are really relevant for our purposes[12]. Although there is a considerable number of terms, there is a very limited number that have a non–zero limit. With $\mathcal{O}$ as in (5.2) one obtains

$$
\begin{aligned}
\lim_{\Lambda^2\to\infty}\Omega_{\mathcal{A}_1} &= \frac{id^2}{6\pi}\lim_{\Lambda^2\to\infty}\mathrm{Tr}\left\{\varepsilon^2\left[u\partial^3\,u\,\mathcal{O}\,\partial^2\,b^*\,\partial - c\,\mathcal{O}\,\partial^2\left(u\,\partial^3\,v^*\,\partial - v^*\,\partial^3\,u\,\partial\right)\right]\right\} \\
&= \frac{d^2}{720\pi^2}\int \mathrm{d}^2x\left[2u(\partial^3 u) - 3(\partial u)(\partial^2 u)\right](\partial^3 b^*) \\
&+ \frac{d^2}{720\pi^2}\int \mathrm{d}^2x(\partial^2 c)\left[2(\partial^4 u)v^* - (\partial^3 u)(\partial v^*) + (\partial u)(\partial^3 v^*) - 2u(\partial^4 v^*)\right].
\end{aligned}
$$

An integration by parts and a glance at the expression for the BRST variation of the two–loop anomaly (5.20) confirms the correctness of the two–loop consistency condition (4.9).

We conclude our illustration of the nonlocally regularized computation of higher loop anomalies and consistency conditions for chiral W_3 Gravity with some remarks. Previous computations of the higher loop anomalies that appeared in the literature have largely rested, explicitly or implicitly, on the assumed W_3 operator product expansion of the quantum theory. For the W_3 case this results in a two–loop anomaly. This was confirmed by a direct computation of the two–loop effective action in [20] using nonlocal regularisation, and by a regularised calculation of the anomaly itself in renormalised Bogoliubov–Parasiuk–Hepp–Zimmerman perturbation theory in [11]. The computation of this section uses, for the first time, a regularized quantum master equation explicitly. The related question, that of consistency conditions for such higher loop anomalies, has been considered previously in ref. [28], using the nonlocal consistency condition (1.10) rather than its local consequence (1.11): it therefore completely differs in spirit from ours, which is aimed at the *local* equations. We have shown explicitly, in the two loop case, how to obtain these local consistency conditions.

[12] The antifield independent term in the expansion (5.22) encodes potential divergencies. They are easily shown to vanish, using $\mathcal{T}(F,n) = \mathrm{Tr}\left[\varepsilon^2(F\partial)_n\right] = 0$ [20], which shows that no counterterms are necessary for this two–loop computation either.

6 Conclusions

The question of the local anomalies emerges in the Field–Antifield or BV formalism as a clash between the formal equations and the locality of the theory, leading to the usual quantum field theory divergences. The problem can be treated either by eliminating the divergences as in the treatment of [11] using Bogoliubov–Parasiuk–Hepp–Zimmerman perturbation theory, or by (temporarily) going to a nonlocal theory through a regularisation. The regularisation in the Pauli–Villars spirit [14] is an example of this last method, but has the disadvantage that it temporarily breaks the gauge invariance (which is afterwards restored by finite counterterms) and that it is mainly suitable for one loop. In this paper, a more drastic nonlocal regularisation scheme [19] was used, which works to all orders and respects gauge invariance, even when considering non–trivial (open, reducible) gauge algebras since it respects the BV structure [20].

The nonlocality of this regularised theory completely eliminates the divergence problem, so that the formal quantum BV master equation becomes a sensible equation, giving an expression for anomalies (at arbitrary order in $\hbar$), and also consistency conditions that they must obey. Removing the cutoff proved to be rather subtle however: it turned out that some contributions to lower loop anomalies that vanish in this limit nevertheless have quantum corrections that do not vanish. This phenomenon was shown to be at the heart of the generation of higher loop anomalies in this scheme.

We have used this mechanism to deduce an explicit expression for the two–loop anomaly and its consistency condition, and shown the viability of the scheme by explicitizing it for W_3 gravity. In particular it was shown that the two loop anomaly satisfies a consistency condition that is *different* from the celebrated equation of Wess and Zumino: it contains an additional inhomogeneous term that is related to the presence of a lower loop anomaly. It is obvious that the method extends perturbatively to higher orders as well.

Since the consistency conditions in higher orders contain inhomogeneous terms, it becomes a non–trivial task to characterise them algebraically. For one loop, it is well known that they are characterised by a local cohomology class. The specific regularisation that one choses reflects on the specific representative of this class that comes out of the actual computation, but one can shift between representatives by adding local finite counterterms to the action. For two loops already a complication arises. In the scheme adopted in this paper, the consistency condition for the two–loop anomaly contains the specific operator representing the one–loop anomaly, in a form that is not obviously cohomological. Stated differently, it depends explicitly on a possible counterterm M_1, which may result in a renormalisation scheme dependence. We leave the algebraic characterisation of these anomalies for the future.

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